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MKN Theory of Bound States

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Abstract

This paper derives the MKN (Maung-Kahana-Norbury) theory of bound states which incorporates the Lande subtraction technique to remove the singularities of the Cornell potential.

1 NRSE in Momentum Space

Non-relativistic Schrödinger equation (NRSE) in configuration space has been solved exactly for some potentials, such as the Coulomb and simple harmonic oscilator potentials. NRSE with a linear potential can be solved analytically for the S-state only as we will show later. For $l > 0$, we resort to numerical methods. NRSE r-space codes are commonly known to be conditionally unstable [1, 2], while the momentum space codes do not have the same problem. The momentum space code has an additional advantage of being easily adaptable to relativistic equations. NRSE in momentum space takes the form

$$\frac{p^2}{2\mu}\,\phi(\mathbf{p}) + \int V(\mathbf{q})\phi(\mathbf{p}')\,dp' = E\,\phi(\mathbf{p}), \tag{1}$$

where $\mathbf{q} = \mathbf{p} - \mathbf{p}'$.
Proof:

NRSE in momentum space can be derived from its configuration space counterpart

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) = E\,\psi(\mathbf{x}) \tag{2}$$

by Fourier transform. First we define the following:

$$\begin{aligned}
\phi(\mathbf{p}) &= \int \psi(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x, && (3)\\
\psi(\mathbf{x}) &= \int \phi(\mathbf{p}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3p, && (4)\\
\mathbf{p} &= \hbar\mathbf{k}, && (5)
\end{aligned}$$

and ignore factors of 2π in Fourier and inverse Fourier transforms. As usual, we assume periodic boundary condition or $\nabla\phi = \phi = 0$ at infinity. We Fourier-transform Eq. [2] term by term. The term on the right hand side of Eq. [2] is obtained simply by Eq. [3]. The first term involves $\nabla^2\psi$ and is transformed as

$$\begin{aligned}
&\int \nabla^2\psi(\mathbf{x})\,e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3x \\
&= \int \nabla\psi(\mathbf{x})\,e^{i\mathbf{k}\cdot\mathbf{x}}\cdot d\mathbf{S} - \int \nabla\psi(\mathbf{x})\cdot\nabla e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3x \\
&= -i\mathbf{k}\cdot\int e^{i\mathbf{k}\cdot\mathbf{x}}\nabla\psi(\mathbf{x})\,d^3x \\
&= -i\mathbf{k}\cdot\left[\int \psi(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}\,d\mathbf{S} - \int \psi(\mathbf{x})\nabla e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3x\right] \\
&= -i\mathbf{k}\cdot\left[-i\mathbf{k}\int \psi(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3x\right] \\
&= -k^2\phi(\mathbf{p}). && (6)
\end{aligned}$$

The second term involves $V(\mathbf{x})\psi(\mathbf{x})$. With Eq. [4], the second term of Eq. [2] is transformed as

$$\begin{aligned}
\int V(\mathbf{x})\psi(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3x &= \int V(\mathbf{x})\left[\int \phi(\mathbf{p}')e^{-i\mathbf{k}'\cdot\mathbf{x}}\,dp'\right]e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3x \\
&= \int\int V(\mathbf{x})\phi(\mathbf{p}')e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\,d^3x\,d^3p' \\
&= \int d^3p'\,\phi(\mathbf{p}')\int d^3x\,V(\mathbf{x})e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\
&= \int V(\mathbf{p}-\mathbf{p}')\phi(\mathbf{p}')\,d^3p' \qquad (7)
\end{aligned}$$

At last, Eqs. [3], [6] and [7] are all needed to Fourier transform Eq. [2] into Eq. [1]. The proof is complete.

In this paper, we attempt to re-derive the results previously obtained by Maung *et al.* [3] in 1993. The power law potential in r space is given by

$$V^N(r) = \begin{cases} 0 & r < 0 \\ \lambda_N \lim_{\eta\to 0} r^N e^{-\eta r} & r \geq 0,\ \eta > 0 \end{cases}$$

Let $G = \hbar = c = 1$. Define $\mathbf{q} \equiv \mathbf{p} - \mathbf{p}'$. The momentum space potential can be obtained by Fourier transform.

$$\begin{aligned}
V^N(\mathbf{q}) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} V^N(r)e^{i\mathbf{r}\cdot\mathbf{q}}d^3r \qquad (8)\\
&= \frac{\lambda_N}{(2\pi)^3}\lim_{\eta\to 0}\int_0^\infty\int_{-1}^{1}\int_0^{2\pi} r^N e^{-\eta r}e^{irq\cos\theta}r^2 dr d\cos\theta d\phi \qquad (9)\\
&= \frac{\lambda_N}{4\pi^2}\frac{1}{iq}\lim_{\eta\to 0}\int_0^\infty\int_{-1}^{1} r^{N+1}e^{-\eta r}e^{irq\cos\theta}dr d(irq\cos\theta) \qquad (10)\\
&= \frac{\lambda_N}{4\pi^2}\frac{1}{iq}\lim_{\eta\to 0}\int_0^\infty r^{N+1}e^{-\eta r}\left(e^{irq}-e^{-irq}\right)dr \qquad (11)\\
&= \frac{\lambda_N}{4\pi^2}\frac{1}{iq}\lim_{\eta\to 0}(-1)^{N+1}\frac{\partial^{N+1}}{\partial\eta^{N+1}}\int_0^\infty e^{-\eta r}\left(e^{irq}-e^{-irq}\right)dr \qquad (12)\\
&= \frac{\lambda_N}{4\pi^2}\frac{1}{iq}\lim_{\eta\to 0}(-1)^{N+1}\frac{\partial^{N+1}}{\partial\eta^{N+1}}\left[\frac{1}{\eta-iq}-\frac{1}{\eta+iq}\right] \qquad (13)\\
&= \frac{\lambda_N}{4\pi^2}\frac{1}{iq}\lim_{\eta\to 0}(-1)^{N+1}\frac{\partial^{N+1}}{\partial\eta^{N+1}}\left[\frac{2iq}{\eta^2+q^2}\right] \qquad (14)
\end{aligned}$$

The final form of the momentum space potential is

$$V^N(\mathbf{q}) = \frac{\lambda_N}{2\pi^2}\lim_{\eta\to 0}(-1)^{N+1}\frac{\partial^{N+1}}{\partial\eta^{N+1}}\left[\frac{1}{\eta^2+q^2}\right], \qquad (15)$$

where $N = -1$ corresponds to the Coulomb potential and $N = 1$ the linear potential. Together they give the Cornell potential

$$V(\mathbf{q}) \equiv V^C(\mathbf{q}) + V^L(\mathbf{q}) = V^{N=-1}(\mathbf{q}) + V^{N=1}(\mathbf{q}). \tag{16}$$

Next we want to perform a partial wave expansion of V^N. There are 3 useful formulas, the Wigner-Eckart Theorem[4]

$$< E'l'm'|T|Elm >= \delta_{l'l}\delta_{m'm}T_l(E), \tag{17}$$

the addition of spherical harmonics

$$\sum_m Y_{lm}(\Omega)Y^*_{lm}(\Omega') = \frac{2l+1}{4\pi}P_l(\cos\theta), \tag{18}$$

and the orthogonality of spherical harmonics

$$\int d\Omega Y^*_{lm}(\Omega)Y_{l'm'}(\Omega) = \delta_{l'l}\delta_{m'm}, \tag{19}$$

which are used in deriving the following result.

$$\begin{aligned}
< \mathbf{p}|V^N|\mathbf{p}' > &= \sum_{lm}\sum_{l'm'} < \mathbf{p}|lm >< lm|V^N|l'm' >< l'm'|\mathbf{p}' > && (20)\\
&= \sum_{lm}\sum_{l'm'} < p\Omega|lm >< lm|V^N|l'm' >< l'm'|p'\Omega' > && (21)\\
&= \sum_{lm}\sum_{l'm'} < p| < \Omega|lm >< lm|V^N|l'm' >< l'm'|\Omega' > |p' > && (22)\\
&= \sum_{lm} < \Omega|lm >< p|V^N_l|p' >< lm|\Omega' > && (23)\\
&= \sum_{lm} V^N_l(p,p')Y_{lm}(\Omega)Y^*_{lm}(\Omega') && (24)\\
&= \sum_{l} \frac{2l+1}{4\pi}V^N_l(p,p')P_l(\cos\theta) && (25)
\end{aligned}$$

In scattering and bound state problems, it is customary to expand the momentum space wavefunction $\phi(\mathbf{p})$ in partial waves, such that

$$\phi(\mathbf{p}) = \sum_{nlm} c_{nlm}\phi_{nl}(p)Y_{lm}(\Omega), \tag{26}$$

where c_{nlm}'s are coefficients of the expansion [p. 396 of Ref 1].

The non-relativistic Schrödinger equation in momentum space is given as

$$\left(\hat{E} - \frac{\mathbf{p}^2}{2\mu}\right)\phi(\mathbf{p}) = \int V^N(\mathbf{q})\phi(\mathbf{p}')d^3\mathbf{p}' \tag{27}$$

$$= \int < \mathbf{p}|V^N|\mathbf{p}' > \phi(\mathbf{p}')d^3\mathbf{p}' \tag{28}$$

Expand NRSE in partial waves.

$$\left(\hat{E} - \frac{\mathbf{p}^2}{2\mu}\right)\sum_{nlm} c_{nlm}\phi_{nl}(p)Y_{lm}(\Omega) = \int p'^2 dp' d\Omega' \sum_{nlm} V_l^N(p,p')Y_{lm}(\Omega)Y^*_{lm}(\Omega')$$

$$\times \sum_{n'l'm'} c_{n'l'm'}\phi_{n'l'}(p')Y_{l'm'}(\Omega') \tag{29}$$

$$= \int p'^2 dp' \sum_{nlm} V_l^N(p,p')c_{nlm}\phi_{nl}(p')Y_{lm}(\Omega). \tag{30}$$

The nl-th terms can be separated by inspection with the help of the identity $\hat{E}\,\phi_{nl}(p) = E_{nl}\,\phi_{nl}(p)$. The partial wave NRSE is

$$\left(E_{nl} - \frac{p^2}{2\mu}\right)\phi_{nl}(p) = \int p'^2 dp' V_l^N(p,p')\phi_{nl}(p'). \tag{31}$$

Use the orthogonality of Legendre polynomials,

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{l'l}, \tag{32}$$

and Eq. (25), we can calculate the potential matrix elements as follow.

$$\int_{-1}^{1} < \mathbf{p}|V^N|\mathbf{p}' > P_l(\cos\theta)d\cos\theta \tag{33}$$

$$= \sum_{l'} \frac{2l'+1}{4\pi} V_{l'}^N(p,p') \int_{-1}^{1} P_l(\cos\theta)P_{l'}(\cos\theta)d\cos\theta \tag{34}$$

$$= \sum_{l'} \frac{1}{2\pi} V_{l'}^N(p,p')\delta_{l'l} \tag{35}$$

$$= \frac{1}{2\pi} V_l^N(p,p') \tag{36}$$

In order words,

$$V_l^N(p,p') = 2\pi \int_{-1}^{1} V^N(\mathbf{q})P_l(\cos\theta)d\cos\theta. \tag{37}$$

Define

$$y \equiv \frac{p^2 + p'^2 + \eta^2}{2p'p}, \tag{38}$$

and use the definition of the Legendre polynomial of the second kind $Q_n(z)$,

$$Q_n(z) = \frac{1}{2}\int_{-1}^{1} \frac{1}{z-t} P_n(t) dt, \tag{39}$$

we can modify Eq. (37) as

$$\begin{aligned}
V_l^N(p,p') &= 2\pi \int_{-1}^{1} V^N(\mathbf{q}) P_l(\cos\theta) d\cos\theta && (40)\\
&= \frac{\lambda_N}{\pi} \lim_{\eta\to 0} \frac{\partial^{N+1}}{\partial\eta^{N+1}} \int_{-1}^{1} \frac{1}{q^2+\eta^2} P_l(\cos\theta) d\cos\theta && (41)\\
&= \frac{\lambda_N}{\pi} \lim_{\eta\to 0} \frac{\partial^{N+1}}{\partial\eta^{N+1}} \int_{-1}^{1} \frac{1}{p^2+p'^2-2p'p\cos\theta+\eta^2} P_l(\cos\theta) d\cos\theta && (42)\\
&= \frac{\lambda_N}{\pi} \lim_{\eta\to 0} \frac{\partial^{N+1}}{\partial\eta^{N+1}} \int_{-1}^{1} \frac{1}{2p'p(y-\cos\theta)} P_l(\cos\theta) d\cos\theta && (43)\\
&= \frac{\lambda_N}{\pi} \lim_{\eta\to 0} \frac{\partial^{N+1}}{\partial\eta^{N+1}} \frac{Q_l(y)}{p'p}. && (44)
\end{aligned}$$

The coulomb potential corresponds to $N = -1$ and has the form

$$V_l^C(p,p') = \frac{\lambda_C}{\pi} \lim_{\eta\to 0} \frac{Q_l(y)}{p'p}. \tag{45}$$

The linear potential corresponds to $N = 1$ and has the form

$$\begin{aligned}
V_l^L &= \frac{\lambda_L}{\pi} \lim_{\eta\to 0} \frac{\partial^2}{\partial\eta^2} \frac{Q_l(y)}{p'p} && (46)\\
&= \frac{\lambda_L}{\pi} \lim_{\eta\to 0} \frac{\partial}{\partial\eta} \left[\frac{\eta}{(p'p)^2} Q_l'(y)\right] && (47)\\
&= \frac{\lambda_L}{\pi} \lim_{\eta\to 0} \left[\frac{Q_l'(y)}{(p'p)^2} + \frac{\eta^2}{(p'p)^3} Q_l''(y)\right]. && (48)
\end{aligned}$$

There are 3 useful relations in terms of Legendre polynomials of the 2nd kind [5]:

$$\begin{aligned}
Q_0(y) &= \frac{1}{2}\ln\left|\frac{y+1}{y-1}\right|, && (49)\\
Q_l(y) &= P_l(y)Q_0(y) - w_{l-1}(y), && (50)\\
w_{l-1}(y) &= \sum_{m=1}^{l} \frac{1}{m} P_{l-m}(y) P_{m-1}(y). && (51)
\end{aligned}$$

The singularities of $V_l^C(p,p')$ and $V_l^L(p,p')$ come from the singularies of $Q_l(y)$ and $Q_l''(y)$. From Eq. (50), it is obvious that the singularities of $Q_l(y)$ and $Q_l''(y)$ again come from those of $Q_0(y)$, $Q_0'(y)$ and $Q_0''(y)$. In order to treat the singularities of the momentum space Cornell potential, we need to control the singularities of $Q_0(y)$, $Q_0'(y)$ and $Q_0''(y)$ first and foremost. Substitute Eq. (38) into Eq. (49), we have

$$Q_0(y) = \frac{1}{2}\ln\left[\frac{(p+p')^2+\eta^2}{(p-p')^2+\eta^2}\right]. \tag{52}$$

Differentiating Eq. (52) yields

$$\begin{aligned} Q_0'(y) &= \frac{1}{2}\frac{\partial}{\partial y}\ln\left|\frac{y+1}{y-1}\right| & (53)\\ &= \frac{1}{2}\left[\frac{1}{y+1}-\frac{1}{y-1}\right] & (54)\\ &= \frac{1}{1-y^2} & (55)\\ &= p'p\left[\frac{1}{(p+p')^2+\eta^2}-\frac{1}{(p-p')^2+\eta^2}\right] & (56)\end{aligned}$$

Differentiating again gives

$$\begin{aligned} Q_0''(y) &= \frac{2y}{(1-y^2)^2} & (57)\\ &= \frac{p^2+p'^2+\eta^2}{p'p}\left[p'p\left(\frac{1}{(p+p')^2+\eta^2}-\frac{1}{(p-p')^2+\eta^2}\right)\right]^2, & (58)\end{aligned}$$

or

$$\frac{\eta^2}{p'p}Q_0''(y) = \eta^2\left(p^2+p'^2+\eta^2\right)\left[\frac{1}{(p+p')^2+\eta^2}-\frac{1}{(p-p')^2+\eta^2}\right]^2. \tag{59}$$

There are two useful identities which we want to prove:

$$\int_0^\infty \frac{1}{p'}Q_0(y,\eta=0)\,dp' = \frac{\pi^2}{2}, \tag{60}$$

$$\int_0^\infty \left[\frac{\eta^2}{p'p}Q_0''(y)+Q_0'(y)\right]dp' = 0. \tag{61}$$

Proof:
The integral in Eq. [60] is derived as follow:

$$\int_0^\infty \frac{1}{p'}Q_0(y,\eta=0)\,dp' \tag{62}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^\infty \frac{1}{p'} \ln\left(\frac{p+p'}{p-p'}\right)^2 dp' && (63)\\
&= \frac{1}{2}\left[\int_0^a \frac{1}{x}\ln\left(\frac{x+a}{x-a}\right)^2 dx + \int_a^\infty \frac{1}{x}\ln\left(\frac{x+a}{x-a}\right)^2 dx\right] && (64)\\
&= \int_a^\infty \frac{1}{x}\ln\left(\frac{a+x}{a-x}\right) dx + \int_a^\infty \frac{1}{x}\ln\left(\frac{x+a}{x-a}\right) dx && (65)\\
&= -\int_\infty^0 \frac{1}{ae^{-u}}\ln\left(\frac{a+ae^{-u}}{a-ae^{-u}}\right) ae^{-u}du + \int_0^\infty \frac{1}{ae^{u}}\ln\left(\frac{ae^{u}+a}{ae^{u}-a}\right) ae^{u}du && (66)\\
&= 2\left[\int_0^\infty \ln\left(\frac{1+e^{-u}}{1-e^{-u}}\right) du\right] && (67)\\
&= 2\left[\int_0^\infty \ln(1+e^{-u})\, du - \int_0^\infty \ln(1-e^{-u})\, du\right] && (68)\\
&= 2\left[\frac{\pi^2}{12}+\frac{\pi^2}{6}\right] && (69)\\
&= \frac{\pi^2}{2} && (70)
\end{aligned}
$$

The results of Eq. [69] come from relations BI((256))(10) and BI((256))(11) in Gradshteyn and Ryzhik [6].

Eq. [56] has 4 simple poles: $\alpha = p+i\eta$, α^*, $\beta = -p+i\eta$ and β^*. $Q_0'(y)$ can be rewritten as

$$
Q_0'(y) = p'p\left[\frac{-1}{(p'-\alpha)(p'-\alpha^*)} + \frac{1}{(p'-\beta)(p'-\beta^*)}\right] \qquad (71)
$$

The contour integral $\oint Q_0'(y)dz$ over the upper complex plane has 2 residues: $Res(\alpha)$ and $Res(\beta)$. Use the formula $Res(z_0) = \lim_{z\to z_0}(z-z_0)f(z)$ to calculate these residues,

$$
\begin{aligned}
Res(\alpha) &= \lim_{p'\to\alpha} p'p\left[\frac{-1}{p'-\alpha^*} + \frac{p'-\alpha}{(p'-\beta)(p'-\beta^*)}\right] && (72)\\
&= -p(p+i\eta)\left[\frac{1}{2i\eta}+0\right] && (73)\\
&= -\frac{p(p+i\eta)}{2i\eta}, && (74)
\end{aligned}
$$

$$
\begin{aligned}
Res(\beta) &= \lim_{p'\to\beta} p'p\left[\frac{-(p'-\beta)}{(p'-\alpha)(p'-\alpha^*)} + \frac{1}{(p'-\beta^*)}\right] && (75)\\
&= p(-p+i\eta)\left[0+\frac{1}{2i\eta}\right] && (76)\\
&= \frac{p(-p+i\eta)}{2i\eta}, && (77)
\end{aligned}
$$

$$\begin{aligned} Res(\alpha) + Res(\beta) &= -\frac{p(p+i\eta)}{2i\eta} + \frac{p(-p+i\eta)}{2i\eta} && (78) \\ &= -\frac{p^2}{i\eta}. && (79) \end{aligned}$$

Since the contour at infinity is zero and $Q_0'(y)$ along the real axis is symmetric around the origin, we obtain

$$\begin{aligned} \int_0^\infty Q_0'(y)\,dp' &= \frac{1}{2}\oint Q_0'(y)\,dz && (80) \\ &= \frac{1}{2}\left(2\pi i \sum Res\right) && (81) \\ &= -\frac{\pi p^2}{\eta}. && (82) \end{aligned}$$

$(\eta^2/p'p)Q_0''(y)$ has the same poles as $Q_0'(y)$ but of order 2. Residues of order m are calculated by the formula

$$Res(z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z). \tag{83}$$

Again we can simplify the algebra by rewriting Eq. [61] as

$$\frac{\eta^2}{p'p}Q_0''(y) = \eta^2\left(p^2+p'^2+\eta^2\right)\left[\frac{-1}{(p'-\alpha)(p'-\alpha^*)} + \frac{1}{(p'-\beta)(p'-\beta^*)}\right]^2. \tag{84}$$

The residues are

$$\begin{aligned} Res(\alpha) &= \lim_{p'\to\alpha}\frac{d}{dp'}\eta^2\left(p^2+p'^2+\eta^2\right)\left[\frac{-1}{(p'-\alpha^*)} + \frac{p'-\alpha}{(p'-\beta)(p'-\beta^*)}\right]^2 && (85) \\ &= \eta^2\left\{2(p+i\eta)\left[\frac{-1}{2i\eta}\right]^2 + 4p(p+i\eta)\left[\frac{-1}{2i\eta}\right]\left[\frac{1}{(2i\eta)^2} + \frac{1}{2p(2p+2i\eta)}\right]\right\} && (86) \\ &= \frac{\eta^2}{2}\left\{-\frac{p}{\eta^2} - \frac{i}{\eta} + \frac{p^2}{i\eta^3} + \frac{p}{\eta^2} - \frac{1}{i\eta}\right\} && (87) \\ &= \frac{p^2}{2i\eta}, && (88) \end{aligned}$$

$$\begin{aligned} Res(\beta) &= \lim_{p'\to\beta}\frac{d}{dp'}\eta^2\left(p^2+p'^2+\eta^2\right)\left[\frac{-(p'-\beta)}{(p'-\alpha)(p'-\alpha^*)} + \frac{1}{p'-\beta^*}\right]^2 && (89) \\ &= \eta^2\left\{2(p-i\eta)\left[\frac{1}{2i\eta}\right]^2 + 4p(p-i\eta)\left[\frac{1}{2i\eta}\right]\left[\frac{-1}{(2i\eta)^2} + \frac{1}{2p(-2p+2i\eta)}\right]\right\} && (90) \end{aligned}$$

$$= \frac{\eta^2}{2}\left\{\frac{p}{\eta^2} - \frac{i}{\eta} + \frac{p^2}{i\eta^3} - \frac{p}{\eta^2} - \frac{1}{i\eta}\right\} \quad (91)$$

$$= \frac{p^2}{2i\eta}. \quad (92)$$

Hence the sum of residues is

$$Res(\alpha) + Res(\beta) = \frac{p^2}{2i\eta} + \frac{p^2}{2i\eta} \quad (93)$$

$$= \frac{p^2}{i\eta} \quad (94)$$

Since $(\eta^2/p'p)Q_0''(y)$ is symmetric around the origin, we can integrate along the same contour as before and obtain

$$\int_0^\infty \frac{\eta^2}{p'p} Q_0''(y)\, dp' = \frac{1}{2}\oint \frac{\eta^2}{pz} Q_0''(y)\, dz \quad (95)$$

$$= \frac{1}{2}\left(2\pi i \sum Res\right) \quad (96)$$

$$= \frac{\pi p^2}{\eta}. \quad (97)$$

From these results, it is obvious that Eq. [61] is true. The proof is complete.

A simple example is the momentum space Schrödinger equation with a linear potential in the S-state [3, 7],

$$\frac{p^2}{2\mu}\,\phi_{n0}(p) + \frac{\lambda_L}{\pi p^2}\int_0^\infty \underbrace{\left[\frac{\eta^2}{p'p} Q_0''(y) + Q_0'(y)\right]}_{V_0^L(p,p')} \phi_{n0}(p')\, dp' = E_{n0}\,\phi_{n0}(p), \quad (98)$$

where $y = (p^2 + p'^2)/2p'p$. Lande subtraction [3, 7] involves subtracting a zero term

$$\int_0^\infty \left[\frac{\eta^2}{p'p} Q_0''(y) + Q_0'(y)\right] dp' = 0 \quad (99)$$

from Eq. [98] such that

$$\frac{p^2}{2\mu}\,\phi_{n0}(p) + \frac{\lambda_L}{\pi p^2}\int_0^\infty \left[\frac{\eta^2}{p'p} Q_0''(y) + Q_0'(y)\right][\phi_{n0}(p') - \phi_{n0}(p)]\, dp' = E_{n0}\,\phi_{n0}(p). \quad (100)$$

Using Eqs. [56,59], the integral in Eq. [100] for $p > 0$ in the limit of $y \to 1$ can be shown to equal

$$\lim_{\eta\to 0}\lim_{p\to p'} \frac{\lambda_L}{\pi}\left[2\eta^2\left(\frac{1}{(p-p')^2+\eta^2}\right)^2 - \frac{1}{(p-p')^2+\eta^2}\right](p-p')^2\,\frac{d\phi_{n0}}{dp} = 0. \quad (101)$$

The order of the limits in Eq. [101] is important. The reverse order will lead to the nonsensical result $\int Q_0'(y)\,dp' = 0$. Next, in the limit of $p, p' \to 0$, $(p+p')^2 = (p-p)^2$. By substituting this equality into Eqs. [56,59], it can be shown again that the integral in Eq. [100] vanishes for $p \to 0$ at $y = 1$. At the end, the integral vanishes at $y = 1$, $\forall\, p$. Away from the singularities, both integrands in the integral of Eq. [100] are finite. By taking $\eta \to 0$, the first integrand vanishes. The final form of Eq. [100] is

$$\frac{p^2}{2\mu}\,\phi_{n0}(p) + \frac{\lambda_L}{\pi p^2}\int_0^\infty Q_0'(y)\,[\phi_{n0}(p') - \phi_{n0}(p)]\,dp' = E_{n0}\,\phi_{n0}(p), \tag{102}$$

where $Q_0'(y) = 1/(1-y^2)$.

The momentum space NRSE with a coulomb potential is given as

$$\frac{p^2}{2\mu}\phi_{nl}(p) + \frac{\lambda_C}{\pi p}\int_0^\infty P_l(y)\frac{Q_0(y)}{p'}\phi_{nl}(p')p'^2dp' - \frac{\lambda_C}{\pi p}\int_0^\infty w_{l-1}(y)\phi_{nl}(p')p'dp' = E_{nl}\phi_{nl}(p). \tag{103}$$

Use Eq. [60] to subtract out the logarithmic singularity and obtain

$$\begin{aligned}\frac{p^2}{2\mu}\phi_{nl}(p) &+ \frac{\lambda_C}{\pi p}\int_0^\infty P_l(y)\frac{Q_0(y)}{p'}\left[p'^2\phi_{nl}(p') - \frac{p^2\phi_{nl}(p)}{P_l(y)}\right]dp' + \frac{\lambda_C}{\pi p}\left[\frac{\pi^2}{2}p^2\phi_{nl}(p)\right] \\ &-\frac{\lambda_C}{\pi p}\int_0^\infty w_{l-1}(y)\phi_{nl}(p')p'dp' = E_{nl}\phi_{nl}(p).\end{aligned} \tag{104}$$

Before we perform Lande subtraction on the NRSE with a confining potential, we need the identity

$$P_l'(1) = \frac{l(l+1)}{2}. \tag{105}$$

Proof:
We use the recursion relation $xP_x' - P_{l-1}'(x) = lP_l(x)$, the equality $P_l(1) = 1$ to obtain the following relations:

$$\begin{aligned} P_l'(1) - P_{l-1}'(1) &= l \\ P_{l-1}'(1) - P_{l-2}'(1) &= l-1 \\ &\vdots \\ P_2'(1) - P_1'(1) &= 2 \\ P_1'(1) - P_0'(1) &= 1. \end{aligned} \tag{106}$$

Add these relations and use $P_0(x) = 1$ or $P_0'(x) = 0$, we prove Eq. [105]. Next we want to examine the singularities of $Q_l'(y)$ and $Q_l''(y)$. Differentiating Eq. [50] once and twice, we have

$$Q_l' = P_l'Q_0 + P_lQ_0' - w_{l-1}', \tag{107}$$

$$Q_l'' = P_l''Q_0 + 2P_l'Q_0' + P_lQ_0'' - w_{l-1}''. \tag{108}$$

$\eta^2 P_l'' Q_0$, $\eta^2 P_l' Q_0'$ and $\eta^2 w_{l-1}''$ vanish in the limit of $\eta \to 0$. The momentum space confining potential is

$$V_l^L(p', p) = \frac{\lambda_L}{\pi} \lim_{\eta \to 0} \left\{ P_l(y) \left[\frac{\eta^2}{(p'p)^3} Q_0''(y) + \frac{Q_0'(y)}{(p'p)^2} \right] + \frac{P_l'(y) Q_0(y) - w_{l-1}'(y)}{(p'p)^2} \right\}. \quad (109)$$

Lastly, perform Lande subtraction and use Eq. [105]. Take the limit of $\eta = 0$, we derive the momentum space NRSE with a confining potential as

$$\begin{aligned}
&\frac{p^2}{2\mu}\phi_{nl}(p) + \frac{\lambda_L}{\pi p^2} \int_0^\infty P_l(y) Q_0'(y) \left[\phi_{nl}(p') - \frac{\phi_{nl}(p)}{P_l(y)} \right] dp' \\
&+ \frac{\lambda_L}{\pi p^2} \int_0^\infty P_l'(y) \frac{Q_0(y)}{p'} \left[p'\phi_{nl}(p') - \frac{l(l+1)}{2} \frac{p\phi_{nl}(p)}{P_l'(y)} \right] dp' \\
&+ \frac{\lambda_L}{\pi p^2} \frac{l(l+1)}{2} \left[\frac{\pi^2}{2} p\phi_{nl}(p) \right] + \frac{\lambda_L}{\pi p^2} \int_0^\infty w_{l-1}'(y) \phi_{nl}(p')\, dp' = E_{nl}\phi_{nl}(p). \quad (110)
\end{aligned}$$

2 Exact Solution of NRSE with a Linear Potential

In the next section, we are going to solve the NRSE numerically with a linear potential. In this section, we will solve the same equation exactly so that we can use the analytic results to check their numerical counterparts. The Hamiltonian equation can be written as

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} \right) R - 2\mu[\lambda_L\, r - E]\, R = 0. \quad (111)$$

Let $S \equiv r\, R$, then Eq. [111] can be simplified as

$$\frac{d^2}{dr^2} S - 2\mu[\lambda_L\, r - E] S = 0. \quad (112)$$

If we define as new variable

$$x \equiv \left(\frac{2\mu^2}{\lambda_L} \right)^{\frac{1}{3}} [\lambda_L r - E], \quad (113)$$

Eq. [113] can be transformed as

$$S'' - xS = 0, \quad (114)$$

which is the Airy equation. The solution which satisfies the boundary condition of $S \to 0$ as $x \to \infty$ is the Airy function $\mathrm{Ai}(x)$. Fig. [1] illustrates the graph of the Airy function. By noticing that $S \equiv r\, R$ vanishes at $r = 0$, we infer that $S(r = 0)$ must coincide with a zero of $\mathrm{Ai}(x)$. It is made possible by letting the eigen-energy act as a horizontal shift which shifts the origin to the left along the x-axis. If S is plotted against r instead of

x, S will vanish at the origin if E_n is chosen appropriately. This conclusion leads to the eigen-energy formula

$$E_n = -x_n \left(\frac{\lambda_L^2}{2\mu}\right)^{\frac{1}{3}}, \tag{115}$$

where x_n is the n-th zero of the Airy function counting from $x = 0$ along $-x$.

In Norbury *et. al.*'s [7] paper, the values $\lambda_L = 5$ and $\mu = 0.75$ are used. The eigen-energy formula is

$$E_n = -2.554364772\, x_n. \tag{116}$$

Table 1 lists the zeros of the Airy function and the corresponding exact eigen-energies.

3 Conclusion

The p-space formalism shown in this paper can be applied to any arbitrary potential in principle although only the cases of $n = -1, 1$ in r^n are considered in this paper. More complicated potentials require the calculation of integrals involving other powers of r. The potential involving the coupling constant $\alpha(r)$ in QCD near the asymptotic freedom region is such an example. In these cases, we may need to reply on numerical integration to evaluate the integrals of the Lande subtraction terms.

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Table 1: Zeros of the Airy function and the corresponding eigen-energies in GeV for $l = 0$, $\mu = 0.75\,\mathrm{GeV}$, $\lambda_L = 5\,\mathrm{GeV}$.

n	x_n	E_n
1	-2.33810741	5.972379202
2	-4.08794944	10.44211404
3	-5.52055983	14.10152355
4	-6.78670809	17.33572806
5	-7.94413359	20.29221499
6	-9.02265085	23.04714148
7	-10.04017434	25.64626764
8	-11.00852430	28.11978666
9	-11.93601556	30.48893766
10	-12.82877675	32.76937540

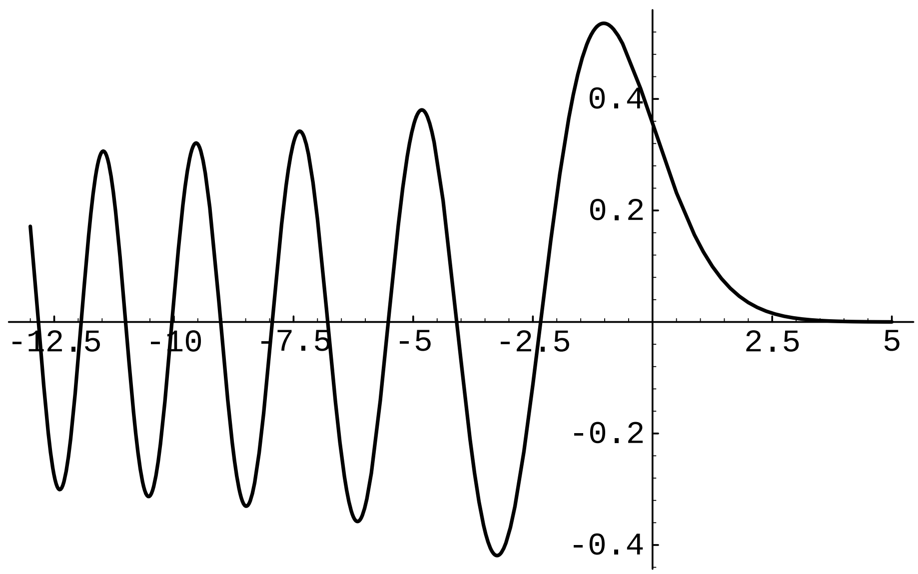

Figure 1: The graph of the Airy function–The zeros are all negative. Since $S = r\,R$, S must be zero at the origin $r = 0$. The eigen-energy acts like a horizontal shift which shifts the origin back to a zero.